EECS 16B Section 8B

W-3/10

Main Topic: Feedback Control

Administrivia:

- HW 8 due Fri, 3/12
- Anonymous Feedback:
  bit.ly/maxwell-16B-feedback-sp21
- Midterm coming up (oof)
  - Staff, CSM/HKN Review Sessions
  - Scope up to 3/4 Lec

Agenda:

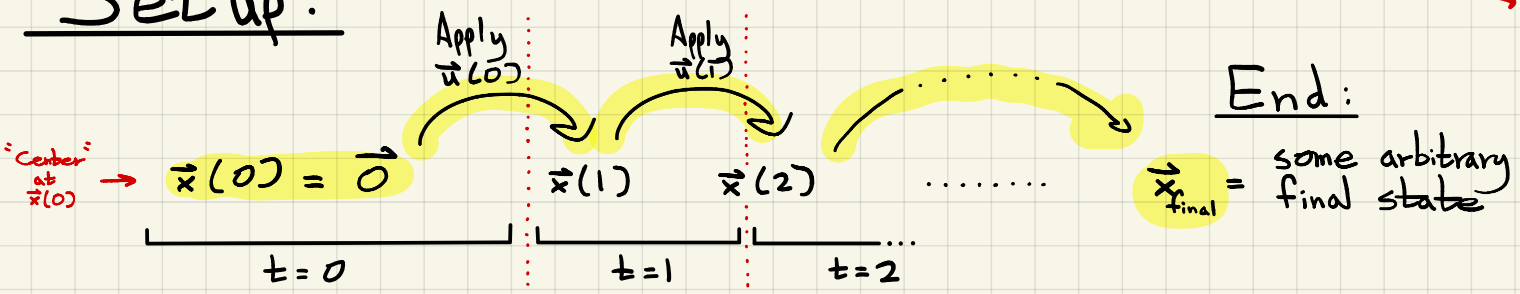
- Controllability
  - Derivation, Controllability Matrix
  - Q3
- Feedback
  - Q1

"Controllable? What's that?"

- "Given a set of inputs, we can get the system from any initial state to any final state"
- "It is POSSIBLE to get from here to there ... ... with the right inputs"

$$\vec{x}[t+1] = A\vec{x}[t] + b\vec{u}[t]$$

Setup:



✓ $t=0$: $\vec{x}(0) = \vec{0}$

✓ $t=1$: $\vec{x}(1) = A\vec{x}(0) + B\vec{u}(0) = A(\vec{0}) + B\vec{u}(0) = B\vec{u}(0)$

"effect of input at t=0 on state at t=1"

$t=2$: $\vec{x}(2) = A\vec{x}(1) + B\vec{u}(1) = A(B\vec{u}(0)) + B\vec{u}(1)$

$$= AB\vec{u}(0) + B\vec{u}(1)$$

$t=3$: $\vec{x}(3) = A(A(B\vec{u}(0)) + B\vec{u}(1)) + B\vec{u}(2)$

$$= A^2B\vec{u}(0) + AB\vec{u}(1) + B\vec{u}(2)$$

⋮

$t=k$: $\vec{x}(k) = A^{k-1}B\vec{u}(0) + A^{k-2}B\vec{u}(1) + \ldots + AB\vec{u}(k-2) + B\vec{u}(k-1)$

- Inputs will have a lingering effect on future states
- Effect of input "travels through time"
  - represented by repeatedly multiplying by A at each time step

Controllability Matrix

$\vec{x}[1] = B\vec{u}[0]$

Interpretation: If we can pick $\vec{u}[0]$, then we can "go" anywhere in the span of B.

$\vec{x}[2] = AB\vec{u}[0] + B\vec{u}[1]$

[Each time step adds a new "degree of freedom"]

Interpretation: If we can pick $\vec{u}[0]$ AND $\vec{u}[1]$, we can go anywhere in the span of $[AB \;\; B]$

$$\vec{x}[3] \longrightarrow \text{Span}\,[A^2B \;\; AB \;\; B]$$

$$\vdots$$

$$\vec{x}[k] \longrightarrow \text{Span}\,[A^{k-1}B \;\; \dots \;\; A^2B \;\; AB \;\; B]$$

Controllability Matrix C

"Mapping" from control inputs to state space

Significance of "C":

- If C is full rank (aka rank n), then our system is controllable

dimension of our state space:

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \updownarrow 2 \Rightarrow n = 2$

(A is $n \times n$ in our State-Space Model)

3. Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[t+1] = \overset{A}{\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \vec{x}[t] + \overset{B}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}} \boxed{u[t]}$$

$$\dim(C) = \dim(A)$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Is the system controllable?

$$\hookrightarrow \text{Construct } C = 3 \times 3 = \left[\underline{A^2B} \quad \underline{AB} \quad B\right]$$

$$\overset{A}{\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}}$$

$$\overset{A}{\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{AB}{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}} = \overset{A^2B}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$= \text{rank } 2$

$2 < 3$, Not controllable

(b) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T - 1$?

$$B =_n \left[\vdots\ \vdots\ \vdots\right]_r \overset{u}{\left[\vdots\right]}$$

$n \times nr$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\vec{x}[1]} = \overset{A}{\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{\vec{x}[0]}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{\vec{B}}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}} u[0]$$

$$= \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2u[0] \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

$$\vec{x}[2] = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}$$

$$\vec{x}[3] = \begin{bmatrix} 8 \\ -15 + 2u[1] \\ -6 + 2u[0] + 2u[2] \end{bmatrix}$$

(c) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T - 1$?

(d) Find the set of all possible states reachable after two timesteps. "at after 2 timesteps"

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ \underline{\underline{2u[0]}} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ \underline{\underline{-2}} \end{bmatrix} \qquad \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$$

$u[0] = -1$

$t = 1$

$$\vec{x}[\underline{2}] = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}$$

We can reach $\underline{\vec{x}[2] = \begin{bmatrix} 4 \\ y \\ z \end{bmatrix} \quad \forall\, y, z}$

$$\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Feedback

Main Principle: Stabilize an Unstable System

How? → Stability is determined by value of λ of the system; so, let's change the λ

D.T.: $|\lambda| < 1$

C.T.: $Re\{\lambda\} < 0$

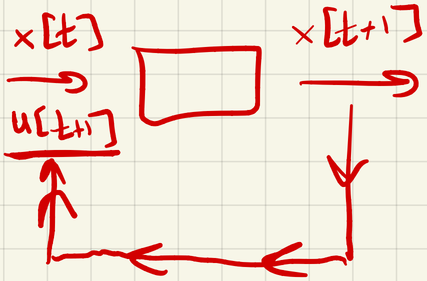
~~Open-Loop:~~ ~~$\vec{u}(t) = ?$~~

Closed-Loop: $\vec{u}(t) = K\vec{x}(t)$

$K \to [k_1 \; k_2 \; \dots]$

So,

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$
$$= A\vec{x}(t) + B(K\vec{x}(t))$$
$$= (A+BK)\vec{x}(t)$$



→ New Matrix; find expression for λ and pick values for k

1. Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

2×2 2×1

C = 2×2

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] + \vec{w}[t] \tag{1}$$

D.T.: $|\lambda| < 1$

(a) Is the system given in eq. (1) stable? Controllable?

Find λ

evec eval

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = 0$

$(A - \lambda I)\vec{v} = 0$

$|A - \lambda I| = 0$

$\left|\begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix}\right| = 0 = \lambda^2 + \lambda - 2$

$0 = (\lambda + 2)(\lambda - 1)$

$\lambda = 1, \boxed{-2}$

Unstable

$C = [B \;\; AB]$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

rank 2 = 2, controllable

(b) Derive a state space representation of the resulting closed loop system using state feedback of the form $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$.

Hint: If you're having trouble parsing this expression for $u[t]$, note that $\begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is a *row vector*, while $\vec{x}[t]$ is a *column vector*. What happens when we multiply a row vector with a column vector like this?

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \;\; k_2]} \vec{x}[t]$$

$(2\times1)(1\times2)$

$= (2\times2)$

= "outer product"

$$\vec{x}[t+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right) \vec{x}[t]$$

$$= \begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix} \vec{x}[t]$$

(c) Find the appropriate state feedback constants, k_1, k_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) Is the system now stable?

$A + Bk$

$$= \begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix} \vec{x}[t]$$

$$\left| \begin{bmatrix} k_1 - \lambda & 1+k_2 \\ 2 & -1-\lambda \end{bmatrix} \right| = 0 = (k_1 - \lambda)(-1-\lambda) - 2(1+k_2)$$

$$0 = -k_1 - k_1\lambda + \lambda + \lambda^2 - 2 - 2k_2$$

$$0 = \lambda^2 + \lambda(1-k_1) - 2k_2 - k_1 - 2$$

$$= (\lambda - \tfrac{1}{2})(\lambda + \tfrac{1}{2})$$

$$= \lambda^2 + 0\lambda - \frac{1}{4}$$

$$\lambda = -\frac{1}{2} \rightarrow (\lambda + \tfrac{1}{2}) = 0$$

$$= \frac{1}{2} \rightarrow (\lambda - \tfrac{1}{2}) = 0$$

$$1 - k_1 = 0$$

$$\rightarrow \boxed{k_1 = 1}$$

$$-2k_2 - 1 - 2 = -\frac{1}{4}$$

$$-2k_2 = 3 - \frac{1}{4}$$

$$-2k_2 = \frac{12}{4} - \frac{1}{4}$$

$$-2k_2 = \frac{11}{4}$$

$$\boxed{k_2 = -\frac{11}{8}}$$

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$ as the way that the discrete-time control acted on the system. As before, we use $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

[General Process for Solving Controllability Questions:]

Discrete-Time System

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

Continuous-Time System

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$A \in \mathbb{R}^{n\times n}$

$\vec{x} \in \mathbb{R}^{n\times 1}$

$\vec{u} \in \mathbb{R}^{1\times 1}$

$B \in \mathbb{R}^{n\times 1}$

Note: Sometimes $\vec{u} \in \mathbb{R}^{r\times 1}$, $B \in \mathbb{R}^{n\times r}$,
so $C = n \times nr$ matrix

$\downarrow$

Calculate $n \times n$ Controllability Matrix C:

$$C = n\left\{\underbrace{\left[\underset{\longleftarrow r \longrightarrow}{A^{n-1}B} \quad \underset{\longleftarrow r \longrightarrow}{A^{n-2}B} \quad \dots \quad \underset{\leftarrow r \rightarrow}{AB} \quad \underset{\leftarrow r \rightarrow}{B}\right]}_{n \text{ "elements", each } r \text{ "wide"}}\right.$$

$[r = 1]$

If C is <u>full rank</u> (i.e. rank $= n$), then the system is <u>controllable</u>

① System in form $\vec{x}[t+1] = A\vec{x}[t] + Bu[t]$ D.T.

or

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$ C.T.

② "Is system controllable?" → Is $C = [B \ AB \ \dots]$ full rank?

③ "Is system stable?" $\xrightarrow{DT}$ $|\lambda_i| < 1$

$\xrightarrow{CT}$ $\text{Re}\{\lambda_i\} < 0$

Controllable	Stable	Result
✓	✓	Fine - Already Stable.
✗	✓	Fine - Already Stable.
✓	✗	Use feedback $u(t) = Kx(t)$ and solve for k given desired λ
✗	✗	Uh oh. Can't control unstable system.

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$$x[0] = \begin{bmatrix} - \\ - \\ - \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x[1] = \begin{bmatrix} - \\ - \\ u[0] \end{bmatrix}$$

$u[0]$

$$x[4] = \begin{bmatrix} u[1] \\ u[2] \\ u[3] \end{bmatrix}$$

$$x[2] = \begin{bmatrix} - \\ u[0] \\ u[1] \end{bmatrix}$$

$$x[3] = \begin{bmatrix} u[0] \\ u[1] \\ u[2] \end{bmatrix}$$